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Algorithms for b-functions, restrictions, and algebraic local cohomology groups of D-modules

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1 Introduction

Let K be an algebraically closed field of characteristic zero and let X be a Zariski open set of K^n with a positive integer n. We fix a coordinate system $x = (x_1, \dots, x_n)$ of X and write $\partial = (\partial_1, \dots, \partial_n)$ with $\partial_i := \partial/\partial x_i$. We denote by $\mathcal{D}_X$ the sheaf of algebraic linear differential operators on X.

Let $\mathcal{M}$ be a coherent left $\mathcal{D}_X$-module and u a section of $\mathcal{M}$. Suppose that $f = f(x) \in K[x]$ is an arbitrary non-constant polynomial of n variables. If $\mathcal{M}$ is holonomic, then for each point p of $Y := \{x \in X \mid f(x) = 0\}$, there exist a germ $P(x, \partial, s)$ of $\mathcal{D}_X[s]$ at p and a polynomial $b(s) \in K[s]$ of one variable so that

$$P(x, \partial, s)(f^{s+1}u) = b(s)f^s u \tag{1.1}$$

holds with an indeterminate s (cf. [8]). More precisely, (1.1) means that there exists a nonnegative integer m so that

$$Q := f^{m-s}(b(s) - P(x, \partial, s)f)f^s \in \mathcal{D}_X[s]$$

satisfies $Qu = 0$ in $\mathcal{M}[s] := K[s] \otimes_K \mathcal{M}$. The monic polynomial $b(s)$ of the least degree that satisfies (1.1), if any, is called the (generalized) *b-function* for f and u at p. The b-function in this sense was first studied by Kashiwara [8] (cf. also [29]). Some of its applications were given by Kashiwara-Kawai [11]. In particular, when $\mathcal{M}$ coincides with the sheaf $\mathcal{O}_X$ of regular functions and $u = 1$, we get the classical b-function (or the Bernstein-Sato polynomial) of f. An algorithm for computing the Bernstein-Sato polynomial has been given in [20].

Suppose that a presentation (i.e., generators and the relations among them) of a coherent left $\mathcal{D}_X$-module $\mathcal{M}$ and a section u of $\mathcal{M}$ are given. Then we are concerned with algorithms for solving the following problems:

(A1) to determine whether there exists and to find, if it does, the b-function for f and u;

(A2) to obtain presentations of the algebraic local cohomology groups $\mathcal{H}^j_{[Y]}(\mathcal{M})$ ($j = 0, 1$) as left $\mathcal{D}_X$-modules (cf. [8] for the definition);

(A3) to obtain a presentation of the localization $\mathcal{M}(*Y) = \mathcal{M}[f^{-1}]$ of $\mathcal{M}$ by f as a left $\mathcal{D}_X$-module;

(A4) to obtain a presentation of the left $\mathcal{D}_X[s]$-module $\sum_{i=1}^r \mathcal{D}_X[s](f^s \otimes u_i)$, where $u_1, \dots, u_r$ are generators of $\mathcal{M}$ and $f^s \otimes u_i$ is regarded as a section of $(\mathcal{O}_X[s, f^{-1}]f^s) \otimes_{\mathcal{O}_X} \mathcal{M}$.

It turns out that these problems are closely related with one another not only from theoretical but also from algorithmic point of view: Solutions to (A2)–(A4) need the existence of and some information on the b-functions for f and $u_1, \dots, u_r$; one can solve the problem (A3) by using a solution to (A4) by specializing the parameter s to an appropriate negative integer. As an application, for two polynomials $f_1, f_2 \in K[s]$, we can obtain a presentation of the left $\mathcal{D}_X$-module $\mathcal{D}_X(f_1^{s_1} f_2^{s_2})$ for generic constants $s_1, s_2 \in K$.

Kashiwara [8] proved that $\mathcal{H}^j_{[Y]}(\mathcal{M})$ and $\mathcal{M}(*Y)$ are holonomic if so is $\mathcal{M}$. In this case (more generally, under a weaker condition that the b-functions for f and $u_1, \dots, u_r$ exist, which can be determined algorithmically), we can solve the problems (A1)–(A4) completely except that we need the condition $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$ to solve the latter part of (A1), (A3), and (A4); even if this condition fails, we can obtain certain information (estimates 'from above') on solutions of these problems. We solve the problem (A4) by generalizing a method developed in [21] for computing a presentation of $\mathcal{D}_X[s]f^s$.

Our algorithms for (A1) and (A2) are actually obtained as applications of algorithms for more general problems as follows: Now let $\mathcal{M}$ be a left coherent $\mathcal{D}_{\tilde{X}}$-module with $\tilde{X} := K \times X$. Let $u_1, \dots, u_r$ be generators of $\mathcal{M}$. We identify X with the hyperplane $\{(t, x) \in \tilde{X} \mid t = 0\}$ of $\tilde{X}$. Then the b-function of $\mathcal{M}$ along X at $p \in X$ is the monic polynomial $b(s) \in K[s]$ of the least degree that satisfies

$$(b(t\partial_t) + tP_i(t, x, t\partial_t, \partial))u_i = 0 \quad (i = 1, \dots, r)$$

with germs $P_i(t, x, t\partial_t, \partial)$ of $\mathcal{D}_{\tilde{X}}$ at p, where we write $\partial_t := \partial/\partial t$. $\mathcal{M}$ is called *specializable* along X at p if such $b(s)$ exists. On the other hand, the *restriction* (also called the induced system or the tangential system) of $\mathcal{M}$ to X is the complex of left $\mathcal{D}_X$-modules:

$$\mathcal{M}^\bullet_X \quad : \quad 0 \longrightarrow \mathcal{M} \xrightarrow{t} \mathcal{M} \longrightarrow 0.$$

It was proved by Laurent-Schapira [13] (and by Kashiwara [8]) that if $\mathcal{M}$ is specializable along X (or holonomic), then the cohomology groups of $\mathcal{M}^\bullet_X$ are coherent left $\mathcal{D}_X$-modules (holonomic systems, respectively).

Assume now that a presentation of a coherent left $\mathcal{D}_{\tilde{X}}$-module $\mathcal{M}$ is given. Then we obtain a complete algorithm for solving the problem

(B1) to determine whether $\mathcal{M}$ is specializable along X and to find, if so, the b-function of $\mathcal{M}$ along X.

This algorithm is obtained by generalizing a method of Gröbner basis computation (the Buchberger algorithm [4]) in the Weyl algebra with respect to the so-called V-filtration ([9]) developed in [18], [19], [20]. We have solved (B1) for the case $r = 1$ in [20]. Here we generalize an algorithm of [20] so that we can compute the b-function as a function of the point of X for arbitrary $r \geq 1$.

Under the condition that $\mathcal{M}$ is specializable along X, we also get an algorithm to solve the problem

(B2) to obtain presentations of the cohomology groups of $\mathcal{M}^\bullet_X$ as left $\mathcal{D}_X$-modules.

It seems that no complete algorithm for (B2) used to be known (see [26],[27],[19] for partial algorithms). Note that $\mathcal{M}$ is specializable if $\mathcal{M}$ is holonomic ([12]). Algorithms for (A1) and (A2) are obtained by applying the algorithms for (B1) and (B2) to the module $(\mathcal{D}_{\widetilde{X}}\delta(t-f(x)))\otimes_{\mathcal{O}_X}\mathcal{M}$ for a given $\mathcal{D}_X$-module $\mathcal{M}$, where $\delta(t-f(x))$ denotes the modulo class of $(t-f(x))^{-1}$ in $\mathcal{O}_{\widetilde{X}}[(t-f(x))^{-1}]$. Thus we can solve (A2) under the condition that $(\mathcal{D}_{\widetilde{X}}\delta(t-f(x)))\otimes_{\mathcal{O}_X}\mathcal{M}$ is specializable along X, and (A1), (A3), (A4) under the additional assumption $\mathcal{H}^0_{[Y]}(\mathcal{M})=0$. We can also show that $(\mathcal{D}_{\widetilde{X}}\delta(t-f(x)))\otimes_{\mathcal{O}_X}\mathcal{M}$ is specializable along X if and only if there exists the b-function for f and each generator of $\mathcal{M}$ in the sense of (1.1).

When $K=\mathbf{C}$, we can consider the problems explained so far with $\mathcal{D}_X$ replaced by the sheaf $\mathcal{D}_X^{\mathrm{an}}$ of *analytic* differential operators. Then our algorithms yield correct solutions also in this analytic case if the left $\mathcal{D}_X^{\mathrm{an}}$-module $\mathcal{M}^{\mathrm{an}}$ in question is written in the form $\mathcal{M}^{\mathrm{an}}=\mathcal{D}_X^{\mathrm{an}}\otimes_{\mathcal{D}_X}\mathcal{M}$ with a coherent $\mathcal{D}_X$-module $\mathcal{M}$ whose presentation is given explicitly.

We have implemented the algorithms in the present paper by using computer algebra systems *Kan* [28] developed by Takayama of Kobe University, and *Risa/Asir* [16] developed by Noro et al. at Fujitsu Laboratories Limited. We use *Kan* for Gröbner basis computation in Weyl algebras, and *Risa/Asir* for Gröbner basis computation, factorization, and primary decomposition in polynomial rings.

2 V-filtration and involutory generators

Let $\widetilde{X}$ be a Zariski open subset of $K\times K^n$ with the coordinate system $(t,x)=(t,x_1,\dots,x_n)$. We denote by $\partial_t=\partial/\partial t$ and $\partial=(\partial_1,\dots,\partial_n)$ the corresponding derivations with $\partial_i=\partial/\partial x_i$. Put $X:=\widetilde{X}\cap(\{0\}\times K^n)$. Then X can be identified with a Zariski open subset of K^n. Let $\mathcal{O}_X$ and $\mathcal{O}_{\widetilde{X}}$ be the sheaves of regular functions on X and on $\widetilde{X}$ respectively. We denote by $\mathcal{D}_{\widetilde{X}}$ and $\mathcal{D}_X$ the sheaves of rings of algebraic linear differential operators on $\widetilde{X}$ and on X respectively. Let $\mathcal{D}_{\widetilde{X}}|_X$ be the sheaf theoretic restriction of $\mathcal{D}_{\widetilde{X}}$ to X. Put $\mathcal{J}_X:=\mathcal{O}_{\widetilde{X}}t$. Then for each integer k we put

$$F_k(\mathcal{D}_{\widetilde{X}}):=\{P\in\mathcal{D}_{\widetilde{X}}|_X\mid P(\mathcal{J}_X)^j\in(\mathcal{J}_X)^{j-k}\quad\text{for any } j\geq 0\}.$$

Let $\mathcal{M}$ be a left coherent $\mathcal{D}_{\widetilde{X}}$-module. We assume that $\mathcal{M}$ has a presentation $\mathcal{M}=(\mathcal{D}_{\widetilde{X}})^r/\mathcal{N}$ on $\widetilde{X}$, where $\mathcal{N}$ is a left $\mathcal{D}_{\widetilde{X}}$-submodule of $(\mathcal{D}_{\widetilde{X}})^r$. Then let us put

$$F_k(\mathcal{N}):=\mathcal{N}\cap F_k(\mathcal{D}_{\widetilde{X}})^r,\qquad F_k(\mathcal{M}):=F_k(\mathcal{D}_{\widetilde{X}})^r/F_k(\mathcal{N})$$

for each integer $k\in\mathbf{Z}$. These are called V-filtrations ([9]). The graded ring and modules associated with these filtrations are defined by

$$\begin{aligned}\mathrm{gr}(\mathcal{D}_{\widetilde{X}}) &:= \bigoplus_{k\in\mathbf{Z}} F_k(\mathcal{D}_{\widetilde{X}})/F_{k-1}(\mathcal{D}_{\widetilde{X}}),\\ \mathrm{gr}(\mathcal{N}) &:= \bigoplus_{k\in\mathbf{Z}} F_k(\mathcal{N})/F_{k-1}(\mathcal{N}),\\ \mathrm{gr}(\mathcal{M}) &:= \bigoplus_{k\in\mathbf{Z}} F_k(\mathcal{M})/F_{k-1}(\mathcal{M}).\end{aligned}$$

Then $\mathrm{gr}(\mathcal{M})$ is a coherent left $\mathrm{gr}(\mathcal{D}_{\tilde{X}})$-module. Note that $\mathrm{gr}(\mathcal{D}_{\tilde{X}})$ is isomorphic to $\mathcal{D}_X[t, \partial_t]$, which consists of the sections of $\mathcal{D}_{\tilde{X}}|_X$ that are polynomials in t.

For a nonzero section P of $(\mathcal{D}_{\tilde{X}})^r|_X$, let $k = \mathrm{ord}_F(P)$ be the minimum $k \in \mathbf{Z}$ such that $P \in F_k(\mathcal{D}_{\tilde{X}})^r$. Then let $\hat{\sigma}(P)$ be the modulo class of P in

$$F_k(\mathcal{D}_{\tilde{X}})^r / F_{k-1}(\mathcal{D}_{\tilde{X}})^r \simeq (\mathcal{D}_X[t\partial_t]S_k)^r,$$

where $S_k := \partial_t^k$ if $k \geq 0$ and $S_k := t^{-k}$ otherwise. Moreover, we define $\psi(P)(s) \in (\mathcal{D}_X[s])^r$ so that $\hat{\sigma}(S_{-k}P) = \psi(P)(t\partial_t)$ holds.

Definition 2.1 Let U be a Zariski open subset of X. A subset $\mathbf{G}$ of $\Gamma(U, \mathcal{N}|_X)$ is called a set of *F-involutory generators* of $\mathcal{N}$ on U if $\mathbf{G}$ generates $\mathcal{N}|_X$ as a left $\mathcal{D}_{\tilde{X}}|_X$-module on U and if $\hat{\sigma}(\mathbf{G}) := \{\hat{\sigma}(P) \mid P \in \mathbf{G}\}$ generates $\mathrm{gr}(\mathcal{N})$ as a left $\mathrm{gr}(\mathcal{D}_{\tilde{X}})$-module.

The following two propositions are immediate consequences of the definitions:

Proposition 2.2 *Let $\mathbf{G} = \{P_1, \ldots, P_m\} \subset \Gamma(U, \mathcal{N}|_X)$ be a set of generators of $\mathcal{N}|_X$ on a Zariski open set $U \subset X$. Then $\mathbf{G}$ is a set of F-involutory generators of $\mathcal{N}$ on U if and only if for an arbitrary nonzero element P of the stalk $\mathcal{N}_p$ of $\mathcal{N}$ at $p \in U$, and for an arbitrary integer j, there exist $Q_1, \ldots, Q_m \in \mathcal{N}_p$ so that $\mathrm{ord}_F(Q_i P_i) \leq \mathrm{ord}_F(P)$ $(i = 1, \ldots, m)$ and*

$$P - Q_1P_1 - \ldots - Q_mP_m \in F_j(\mathcal{D}_{\tilde{X}})^r_p.$$

Proposition 2.3 *Let $\mathbf{G}$ be a set of F-involutory generators of $\mathcal{N}$. Denote by $\psi(\mathcal{N})$ the left $\mathcal{D}_X[s]$-submodule of $(\mathcal{D}_X[s])^r$ generated by $\{\psi(P) \mid P \in \mathcal{N}\}$. Then $\psi(\mathcal{N})$ is generated by $\psi(\mathbf{G}) := \{\psi(P) \mid P \in \mathbf{G}\}$.*

3 Gröbner bases with respect to the V-filtration

The purpose of this section is to show that a set of F-involutory generators of a given submodule $\mathcal{N}$ of $(\mathcal{D}_{\tilde{X}})^r$ can be provided by a Gröbner basis in the Weyl algebra with respect to an appropriate term ordering, which can be computed by the Buchberger algorithm ([4]) for Gröbner bases of polynomial rings. The fact that the Buchberger algorithm applies to the Weyl algebra (the ring of differential operators with polynomial coefficients) was observed by Galligo [5] (cf. also [3],[25]).

Let us denote by A_n and A_{n+1} the Weyl algebras on the n variables x and on the $n+1$ variables (t, x) respectively with coefficients in K (cf. [1]). Let r be a positive integer and put $L := \mathbf{N}^{2+2n} = \mathbf{N} \times \mathbf{N} \times \mathbf{N}^n \times \mathbf{N}^n$ with $\mathbf{N} := \{0, 1, 2, \ldots\}$. An element P of $(A_{n+1})^r$ is written in a finite sum

$$P = \sum_{i=1}^{r} \sum_{(\mu,\nu,\alpha,\beta)\in L} a_{\mu\nu\alpha\beta i} t^\mu x^\alpha \partial_t{}^\nu \partial^\beta e_i \tag{3.1}$$

with $a_{\mu\nu\alpha\beta i} \in K$, $e_1 := (1, 0, \ldots, 0), \ldots, e_r := (0, \ldots, 0, 1)$, $x^\alpha := x_1{}^{\alpha_1} \cdots x_n{}^{\alpha_n}$, $\partial^\beta := \partial_1{}^{\beta_1} \cdots \partial_n{}^{\beta_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}^n$.

Let $\prec_F$ be a total order on $L \times \{1, \ldots, r\}$ which satisfies

(O-1) $(\alpha, i) \prec_F (\beta, j)$ implies $(\alpha + \gamma, i) \prec_F (\beta + \gamma, j)$ for any $\alpha, \beta, \gamma \in L$ and $i, j \in \{1, \ldots, r\}$;

(O-2) if $\nu - \mu < \nu' - \mu'$, then $(\mu, \nu, \alpha, \beta, i) \prec_F (\mu', \nu', \alpha', \beta', j)$ for any $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$, $\mu, \nu, \mu', \nu' \in \mathbf{N}$ and any $i, j \in \{1, \dots, r\}$;

(O-3) $(\mu, \mu, \alpha, \beta, i) \succeq_F (0, 0, 0, 0, i)$ for any $\mu \in \mathbf{N}$, $\alpha, \beta \in \mathbf{N}^n$, $i \in \{1, \dots, r\}$.

Note that $\prec_F$ is not a well order (linear ordering). However, throughout the present paper, every order is supposed to satisfy (O-1). Let P be a nonzero element of $(A_{n+1})^r$ which is written in the form (3.1). Then the *leading exponent* $\mathrm{lexp}_F(P) \in L \times \{1, \dots, r\}$ of P with respect to $\prec_F$ is defined as the maximum element

$$\max \{(\mu, \nu, \alpha, \beta, i) \mid a_{\mu\nu\alpha\beta i} \neq 0\}$$

with respect to the order $\prec_F$. The set of leading exponents $E_F(N)$ of a subset N of $(A_{n+1})^r$ is defined by

$$E_F(N) := \{\mathrm{lexp}(P) \mid P \in N \setminus \{0\}\}.$$

Definition 3.1 A finite set $\mathbf{G}$ of generators of a left A_{n+1}-submodule N of $(A_{n+1})^r$ is called an FW-*Gröbner basis* of N if we have

$$E_F(N) = \bigcup_{P \in \mathbf{G}} (\mathrm{lexp}(P) + L),$$

where we write

$$(\alpha, i) + L = \{(\alpha + \beta, i) \mid \beta \in L\}$$

for $\alpha \in L$ and $i \in \{1, \dots, r\}$.

Proposition 3.2 *Let* $\mathbf{G}$ *be an FW-Gröbner basis of a left* A_{n+1}*-submodule* N *of* $(A_{n+1})^r$*. Then* $\mathbf{G}$ *is a set of F-involutory generators of the left* $\mathcal{D}_{\tilde{X}}$*-submodule* $\mathcal{N} := \mathcal{D}_{\tilde{X}} N$ *of* $(\mathcal{D}_{\tilde{X}})^r$ *on* X.

Since the order $\prec_F$ is not a well-order, the Buchberger algorithm for computing Gröbner bases does not work directly. We use the homogenization with respect to the V-filtration in order to bypass this difficulty (cf. [18], [19], [20]). The following arguments generalize those in [20], where the case with $r = 1$ is treated. Since this generalization is straightforward, we omit the proof.

Definition 3.3 For $\lambda, \mu, \nu, \lambda', \mu', \nu' \in \mathbf{N}$, $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$, and $i, j \in \{1, \dots, r\}$, an order $\prec_H$ on $L_1 \times \{1, \dots, r\}$ with $L_1 := \mathbf{N} \times L$ is defined so that we have $(\lambda, \mu, \nu, \alpha, \beta, i) \prec_H (\lambda', \mu', \nu', \alpha', \beta', j)$ if and only if one of the following conditions holds:

(1) $\lambda < \lambda'$;

(2) $\lambda = \lambda'$, $(\mu + \ell, \nu, \alpha, \beta, i) \prec_F (\mu' + \ell', \nu', \alpha', \beta', j)$ with $\ell, \ell' \in \mathbf{N}$ such that $\nu - \mu - \ell = \nu' - \mu' - \ell'$;

(3) $(\lambda, \nu, \alpha, \beta, i) = (\lambda', \nu', \alpha', \beta', j)$, $\mu < \mu'$

This definition is independent of the choice of ℓ, ℓ' in view of the condition (O-1).

For a nonzero element $P = P(x_0)$ of $(A_{n+1}[x_0])^r$, let us denote by $\mathrm{lexp}_H(P) \in L_1 \times \{1, \dots, r\}$ the leading exponent of P with respect to $\prec_H$.

Definition 3.4 An element P of $(A_{n+1}[x_0])^r$ of the form

$$P = \sum_{i=1}^{r} \sum_{\lambda,\mu,\nu,\alpha,\beta} a_{\lambda\mu\nu\alpha\beta i} {x_0}^{\lambda} t^{\mu} x^{\alpha} {\partial_t}^{\nu} \partial^{\beta} e_i$$

is said to be *F-homogeneous* of order m if $a_{\lambda\mu\nu\alpha\beta i} = 0$ whenever $\nu - \mu - \lambda \neq m$.

Definition 3.5 For an element P of $(A_{n+1})^r$ of the form (3.1), put $m := \min\{\nu - \mu \mid a_{\mu\nu\alpha\beta i} \neq 0$ for some $\mu, \nu \in \mathbf{N}$, $\alpha, \beta \in \mathbf{N}^n$, and $i \in \{1, \dots, r\}\}$. Then the *F-homogenization* $P^h \in (A_{n+1}[x_0])^r$ of P is defined by

$$P^h := \sum_{i=1}^{r} \sum_{\mu,\nu,\alpha,\beta} a_{\mu\nu\alpha\beta i} {x_0}^{\nu-\mu-m} t^{\mu} x^{\alpha} {\partial_t}^{\nu} \partial^{\beta} e_i$$

with a parameter x_0 which commutes with all the other variables and derivations. P^h is F-homogeneous of order m.

Proposition 3.6 *Let $\widetilde{N}$ be a left $A_{n+1}[x_0]$-submodule of $(A_{n+1}[x_0])^r$ generated by F-homogeneous operators. Then there exists an H-Gröbner basis (i.e. a Gröbner basis with respect to $\prec_H$) of $\widetilde{N}$ consisting of F-homogeneous operators. Moreover, such an H-Gröbner basis can be computed by the Buchberger algorithm.*

Proposition 3.7 *Let N be a left A_{n+1}-submodule of $(A_{n+1})^r$ generated by $P_1, \dots, P_d \in (A_{n+1})^r$. Let us denote by N^h the left $A_{n+1}[x_0]$-submodule of $(A_{n+1}[x_0])^r$ generated by $(P_1)^h, \dots, (P_d)^h$. Let $\mathbf{G} = \{Q_1(x_0), \dots, Q_k(x_0)\}$ be an H-Gröbner basis of N^h consisting of F-homogeneous operators. Then $\mathbf{G}(1) := \{Q_1(1), \dots, Q_k(1)\}$ is an FW-Gröbner basis of N.*

These two propositions, combined with Proposition 3.2, provide us with an algorithm of computing a finite set of F-involutory generators of $\mathcal{N} = \mathcal{D}_{\widetilde{X}} N$ on X.

4 The b-function of a D-module

We retain the notation in the preceding section. Let $\mathcal{M}$ be a left coherent $\mathcal{D}_{\widetilde{X}}$-module on $\widetilde{X}$. We assume that a left A_{n+1}-submodule N of $(A_{n+1})^r$ is given explicitly so that $\mathcal{M} = \mathcal{D}_{\widetilde{X}} \otimes_{A_{n+1}} M$ holds with $M := (A_{n+1})^r/N$. Set $\mathcal{N} := \mathcal{D}_{\widetilde{X}} \otimes_{A_{n+1}} N \subset (\mathcal{D}_{\widetilde{X}})^r$. Let $F_k(\mathcal{N})$, $F_k(\mathcal{M})$ be the V-filtrations of $\mathcal{N}$ and $\mathcal{M}$ respectively defined in Section 2 and put

$$\begin{aligned} \mathrm{gr}_k(\mathcal{D}_{\widetilde{X}}) &:= F_k(\mathcal{D}_{\widetilde{X}})/F_{k-1}(\mathcal{D}_{\widetilde{X}}), \\ \mathrm{gr}_k(\mathcal{N}) &:= F_k(\mathcal{N})/F_{k-1}(\mathcal{N}), \\ \mathrm{gr}_k(\mathcal{M}) &:= F_k(\mathcal{M})/F_{k-1}(\mathcal{M}). \end{aligned}$$

In particular, $\mathrm{gr}_0(\mathcal{M})$ and $\mathrm{gr}_0(\mathcal{N})$ are left $\mathrm{gr}_0(\mathcal{D}_{\widetilde{X}})$-modules and we can identify $\mathrm{gr}_0(\mathcal{D}_{\widetilde{X}})$ with $\mathcal{D}_X[t\partial_t]$.

Definition 4.1 The *b-function* $b(s,p) \in K[s]$ of $\mathcal{M}$ along X (with respect to the V-filtration $\{F_k(\mathcal{M})\}$) at $p \in X$ is the monic polynomial $b(s,p) \in K[s]$ of the least degree, if any, that satisfies

$$b(t\partial_t, p)\mathrm{gr}_0(\mathcal{M})_p = 0. \tag{4.1}$$

If such $b(s,p)$ exists, $\mathcal{M}$ is called *specializable* along X at p. If $\mathcal{M}$ is not specializable at p, we put $b(s,p) = 0$.

It is known that if $\mathcal{M}$ is holonomic, then $\mathcal{M}$ is specializable at any $p \in X$ ([12]). In the sequel, we describe an algorithm for computing $b(s,p) \in K[s]$ as a function of $p \in X$.

Proposition 4.2 *Put* $\mathcal{J} := \psi(\mathcal{N}) \cap (\mathcal{O}_X[s])^r$*, which is an* $\mathcal{O}_X[s]$*-submodule of* $(\mathcal{O}_X[s])^r$*. Let* $\mathrm{Ann}((\mathcal{O}_X[s])^r/\mathcal{J}) \subset \mathcal{O}_X[s]$ *be the annihilator ideal for* $(\mathcal{O}_X[s])^r/\mathcal{J}$*. Then the ideal* $\mathrm{Ann}((\mathcal{O}_X[s])^r/\mathcal{J})_p \cap K[s]$ *of* $K[s]$ *is generated by* $b(s,p)$ *for each* $p \in X$*.*

A set of generators of $\psi(\mathcal{N})$ on X can be computed by using Propositions 2.3, 3.6, 3,7. Hence our first task here is to compute a set of generators of $\mathcal{J}$. Let $\prec_D$ be a total order on $L_0 \times \{1,\dots,r\}$ with $L_0 := \mathbf{N}^{1+2n}$ which satisfies (O-1) with L replaced by L_0 and

(O-4) $(\alpha, i) \succ_D (0, i)$ for any $\alpha \in L_0 \setminus \{0\}$ and $i \in \{1,\dots,r\}$;

(O-5) $|\beta| < |\beta'|$ implies $(\mu,\alpha,\beta,i) \prec_D (\mu',\alpha',\beta',j)$ for any $\mu,\mu' \in \mathbf{N}$, $\alpha,\alpha',\beta,\beta' \in \mathbf{N}^n$, $i,j \in \{1,\dots,r\}$.

Note that the order $\prec_D$ is a well-order.

Proposition 4.3 *Let* $\mathbf{G}_1$ *be a finite subset of* $(A_n[s])^r$ *which generates* $\psi(\mathcal{N})$ *as a left* $\mathcal{D}_X[s]$*-module on* X*. Let* $\mathbf{G}_2$ *be a Gröbner basis with respect to* $\prec_D$ *of the submodule of* $(A_n[s])^r$ *generated by* $\mathbf{G}_1$*. Put* $\mathbf{G}_3 := \mathbf{G}_2 \cap K[s,x]^r$*. Then* $\mathcal{J}$ *is generated by* $\mathbf{G}_3$ *on* X *as an* $\mathcal{O}_X[s]$*-module.*

The final step will be devoted to the computation of $b(s,p)$ with a set of generators of $\mathcal{J}$ as an input. For $i = 1,\dots,r$, put

$$\mathcal{J}^{(i)} := \{f = (f_1,\dots,f_r) \in \mathcal{J} \mid f_j = 0 \text{ if } j > i\}.$$

Then $\mathcal{J}^{(i)}/\mathcal{J}^{(i-1)}$ can be regarded as an ideal of $\mathcal{O}_X[s]$ whose generators can be computed via a Gröbner basis with respect to an order $\prec$ on $\mathbf{N}^{1+n} \times \{1,\dots,r\}$ satisfying $(\alpha,i) \prec (\beta,j)$ for any $\alpha,\beta \in \mathbf{N}^{1+n}$ if $i < j$.

So far we have used only the Buchberger algorithm, which does not require field extension, for computing Gröbner bases with respect to various orders. Hence we do not need to assume that K is algebraically closed from the viewpoint of algorithms. Thus, in the rest of this section, we assume that K is an arbitrary field of characteristic zero so that the inputs are defined over K. Since we will make use of primary decomposition, which is sensitive to field extension, we will have to pay attention to the coefficient fields.

Let $\overline{K}$ be the algebraic closure of K and suppose that X is a Zariski open subset of $\overline{K}^n$. We denote by $\mathcal{O}_X$ the sheaf of regular functions on X. In particular, $\mathcal{O}_X$ is a sheaf of $\overline{K}$-algebras.

In general, for an ideal Q of $K[s,x]$ and $p \in \overline{K}^n$, let us denote by $b(s,Q,p) \in K[s]$ a generator of the ideal $K[s] \cap \mathcal{O}_X[s]_p Q$. We may assume that $b(s,Q,p)$ is monic if it is not zero. Put

$$\mathbf{V}_X(Q) := \{x \in X \mid f(x) = 0 \quad \text{for any } f \in Q \cap K[x]\}.$$

Note that $\mathbf{V}_X(Q)$ can be computed by eliminating s by means of a Gröbner basis of Q.

Lemma 4.4 *In the above notation, the ideal $\mathcal{O}_X[s]_p Q \cap \overline{K}[s]$ of $\overline{K}[s]$ is also generated by $b(s,Q,p)$.*

Proposition 4.5 *Assume that Q is a primary ideal of $K[s,x]$ and let $h(s,Q)$ be a generator of the ideal $Q \cap K[s]$ of $K[s]$.*

(1) Case $h(s,Q) \neq 0$: In this case there exists an irreducible polynomial $h_0(s,Q) \in K[s]$ and $\nu_0 \in \mathbf{N}$ so that $h(s,Q) = h_0(s,Q)^{\nu_0}$. Put

$$\mathbf{V}_X^{\nu}(Q) := \{x \in X \mid f(x) = 0 \quad \text{for any } f \in K[x] \cap (Q : h_0(s,Q)^{\nu})\}$$

for each $\nu \in \mathbf{N}$, where $:$ denotes the ideal quotient in $K[s,x]$. Then we have a decreasing sequence of algebraic sets

$$X \supset \mathbf{V}_X(Q) = \mathbf{V}_X^0(Q) \supset \mathbf{V}_X^1(Q) \supset \ldots \supset \mathbf{V}_X^{\nu_0}(Q) = \emptyset$$

of X. If $p \in \mathbf{V}_X^{\nu-1}(Q) \setminus \mathbf{V}_X^{\nu}(Q)$, then we have $b(s,Q,p) = h_0(s,Q)^{\nu}$ for $\nu = 0, \ldots, \nu_0$, where we put $\mathbf{V}_X^{-1}(Q) := X$.

(2) Case $h(s,Q) = 0$: In this case we have $b(s,Q,p) = 0$ if $p \in \mathbf{V}_X(Q)$ and $b(s,Q,p) = 1$ otherwise.

Note that $h(s,Q)$ and the ideal quotient $Q : h_0(s,Q)^{\nu}$ can be computed also by Gröbner bases ([4]).

Proposition 4.6 *Under the above assumptions and notation, let J_i be an ideal of $K[s,x]$ such that $\mathcal{O}_X[s]J_i = \mathcal{J}^{(i)}/\mathcal{J}^{(i-1)}$ for $i = 1, \ldots, r$. Let*

$$J_i = Q_{i,1} \cap \ldots \cap Q_{i,m_i}$$

be a primary decomposition of J_i in $K[s,x]$. Then the b-function $b(s,p)$ of $\mathcal{M}$ at $p \in X$ is the least common multiple of $b(s,Q_{i,j},p)$'s where (i,j) runs over the set $\{(i,j) \mid 1 \leq i \leq r,\ 1 \leq j \leq m_i\}$.

Thus by combining Propositions 4.2, 4.3, 4.5 and 4.6, we have obtained an algorithm to compute the b-function $b(s,p)$ of $\mathcal{M}$ as a function of $p \in X$. In particular, note that $b(s,p)$ belongs to $K[s]$ for any $p \in X$. Let us assume that X is defined over K, i.e., there exists an ideal I_X of $K[x]$ so that $\overline{K}^n \setminus X$ is the set of the zeros of I_X in $\overline{K}^n$. Then the following theorem provides us with an algorithm to determine whether $\mathcal{M}$ is specializable along X at *every* point of $p \in X$, and to compute the set $\{s \in \overline{K} \mid b(s,p) = 0 \quad \text{for some } p \in X\}$. This will be needed in order to compute the restriction and the algebraic local cohomology groups globally on X in the subsequent sections (cf. Proposition 5.2 below). Let us denote by rad Q' the radical of an ideal $Q' \subset K[x]$.

Theorem 4.7 *Let J_i and Q_{ij} be as in the preceding proposition.*

(1) $\mathcal{M}$ is specializable along X at each point of X if and only if the condition

$$Q_{ij} \cap K[s] \neq \{0\} \quad \textit{or} \quad \mathrm{rad}(Q_{ij} \cap K[x]) \supset I_X \tag{4.2}$$

holds for each $i = 1, \ldots, r$ and $j = 1, \ldots, m_i$.

(2) Assume that (4.2) holds for each i and j. Let $b_{ij}(s)$ be a generator of $Q_{ij} \cap K[s]$ if $\mathrm{rad}(Q_{ij} \cap K[x]) \not\supset I_X$*, and put $b_{ij}(s) := 1$ if* $\mathrm{rad}(Q_{ij} \cap K[x]) \supset I_X$*. Let $b(s)$ be the least common multiple of $b_{ij}(s)$'s with $1 \leq i \leq r$ and $1 \leq j \leq m_i$. Then the b-function $b(s,p)$ of $\mathcal{M}$ divides $b(s)$ for any $p \in X$. Moreover, for any irreducible factor $g(s)$ of $b(s)$, there exists some $p \in X$ so that $g(s)$ divides $b(s,p)$.*

(3) Assume $X = \overline{K}^n$. Then $\mathcal{M}$ is specializable along X at each point of X if and only if $J_i \cap K[s] \neq 0$ for any $i = 1, \ldots, r$. In this case let $b_i(s)$ be a generator of $J_i \cap K[s]$ and let $b(s)$ be the least common multiple of $b_1(s), \ldots, b_r(s)$. Then $b(s)$ is the least common multiple of $b(s,p)$'s where p runs over X.

5 The restriction of a D-module

We retain the notation of the preceding section. In particular, let $b(s,p)$ be the b-function of $\mathcal{M}$ at $p \in X$. The (D-module theoretic) *restriction* of $\mathcal{M}$ to X is the complex

$$\mathcal{M}_X^\bullet \quad : \quad 0 \longrightarrow \mathcal{M} \xrightarrow{t} \mathcal{M} \longrightarrow 0$$

of left $\mathcal{D}_X$-modules, where the homomorphism t denotes the one defined by $t(u) = tu$ for each $u \in \mathcal{M}$. We regard the right $\mathcal{M}$ to be placed at the degree 0 in considering the cohomology groups of $\mathcal{M}_X^\bullet$. Put $\mathcal{D}_{X \to \tilde{X}} := \mathcal{D}_{\tilde{X}}/t\mathcal{D}_{\tilde{X}}$. Then $\mathcal{D}_{X \to \tilde{X}}$ is a $(\mathcal{D}_X, \mathcal{D}_{\tilde{X}})$-bimodule, and $\mathcal{M}_X^\bullet$ is isomorphic to $\mathcal{D}_{X \to \tilde{X}} \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_{\tilde{X}}} \mathcal{M}$ in the derived category, where $\overset{\mathbf{L}}{\otimes}$ denotes the left derived functor of $\otimes$ (cf. [6]). Let us denote by $\mathcal{M}_X := \mathcal{H}^0(\mathcal{M}_X^\bullet) = \mathcal{M}/t\mathcal{M}$ the 0-th cohomology group of the complex $\mathcal{M}_X^\bullet$.

Lemma 5.1 *The homomorphism $t : \mathrm{gr}_{k+1}(\mathcal{M})_p \longrightarrow \mathrm{gr}_k(\mathcal{M})_p$ is bijective if $b(k,p) \neq 0$ for $p \in X$.*

Proposition 5.2 *Assume that $\mathcal{M}$ is specializable along X at each point of X. Let $k_0 \leq k_1$ be integers such that the b-function $b(s,p)$ of $\mathcal{M}$ satisfies $b(k,p) \neq 0$ for any $p \in X$ and for any integer k such that $k < k_0$ or $k > k_1$. Then $\mathcal{M}_X^\bullet$ is quasi-isomorphic to the complex*

$$0 \longrightarrow F_{k_1+1}(\mathcal{M})/F_{k_0}(\mathcal{M}) \xrightarrow{t} F_{k_1}(\mathcal{M})/F_{k_0-1}(\mathcal{M}) \longrightarrow 0$$

of left $\mathcal{D}_X$-modules on X. In particular, $t : \mathcal{M} \longrightarrow \mathcal{M}$ is bijective if $b(k,p) \neq 0$ for any $p \in X$ and $k \in \mathbf{Z}$.

The following proposition provides a sufficient condition for the -1th cohomology group $\mathcal{H}^{-1}(\mathcal{M}_X^\bullet)$ to vanish.

Proposition 5.3 *Assume that there exists $b_0(s) \in K[s]$ and $m \in \mathbf{N}$ so that*

$$b_0(t\partial_t){\partial_t}^m \mathrm{gr}_0(\mathcal{M})_p = 0.$$

Assume, moreover, $b_0(k) \neq 0$ for any $k \in \mathbf{Z}$. Then the homomorphism $t : \mathcal{M}_p \longrightarrow \mathcal{M}_p$ is injective.

Now we shall give an algorithm to compute $\mathcal{M}_X$. Let P be an element of $F_m(\mathcal{D}_{\widetilde{X}})^r$. Then we can write P in the form

$$P = \sum_{i=1}^{r} \sum_{k=0}^{m} P_{ik}(t\partial_t, x, \partial){\partial_t}^k e_i + R$$

uniquely with $P_{ik} \in \mathcal{D}_X[t\partial_t]$ and $R \in F_{-1}(\mathcal{D}_{\widetilde{X}})^r$. Then we put

$$\rho(P, k_0) := \sum_{i=1}^{r} \sum_{k=k_0}^{m} P_{ik}(0, x, \partial){\partial_t}^k e_i$$

for each integer k_0 with $0 \leq k_0 \leq m$.

Theorem 5.4 *Assume that $\mathcal{M}$ is specializable along X and let k_0, k_1 be as in Proposition 5.2. Redefine k_0 to be 0 if $k_0 < 0$. (We have $k_0 = 0$ and $k_1 = m - 1$ under the assumption of Proposition 5.3.) Let $\mathbf{G}$ be a finite set of F-involutory generators of $\mathcal{N}$ on X. Then we have an isomorphism*

$$\mathcal{M}_X \quad \simeq \quad (\bigoplus_{i=1}^{r} \bigoplus_{k=k_0}^{k_1} \mathcal{D}_X {\partial_t}^k e_i)/\mathcal{N}_X$$

of left $\mathcal{D}_X$-modules, where $\mathcal{N}_X$ is the left $\mathcal{D}_X$-module generated by a finite set

$$\mathbf{G}_X := \{\rho({\partial_t}^j P, k_0) \mid P \in \mathbf{G},\ j \in \mathbf{N},\ k_0 \leq j + \mathrm{ord}_F(P) \leq k_1\}.$$

In particular, we have $\mathcal{M}_X = 0$ if $b(\nu, p) \neq 0$ for any $\nu \in \mathbf{N}$ and $p \in X$.

In order to interpret the preceding theorem more concretely, let $u_1, \ldots, u_r$ be the modulo classes of $e_1, \ldots, e_r$ in $\mathcal{M}$. Then as is seen by the proof of the preceding theorem, $\mathcal{M}_X \simeq \mathcal{D}_{X \to \widetilde{X}} \otimes_{\mathcal{D}_{\widetilde{X}}} \mathcal{M}$ is generated by $1 \otimes ({\partial_t}^k u_i)$ with $k_0 \leq k \leq k_1$ and $1 \leq i \leq r$ as left $\mathcal{D}_X$-module. Moreover, for $P_{ik} \in \mathcal{D}_X$, we have

$$\sum_{i=1}^{r} \sum_{k=k_0}^{k_1} P_{ik}(1 \otimes {\partial_t}^k u_i) = 0$$

if and only if $\sum_{i=1}^{r} \sum_{k=k_0}^{k_1} P_{ik} e_i \in \mathcal{N}_X$.

Our next aim is to give an algorithm for computing the structure of the kernel $\mathcal{H}^{-1}(\mathcal{M}^\bullet_X)$ of $t : \mathcal{M} \longrightarrow \mathcal{M}$ as a left $\mathcal{D}_X$-module. Note that $\mathcal{H}^{-1}(\mathcal{M}^\bullet_X)$ has a structure of left $\mathcal{D}_X[t\partial_t]$-module which is compatible with that of left $\mathcal{D}_X$-module. For two integers $k_0 \leq k_1$, put

$$\widetilde{\mathcal{D}}^{(k_0,k_1)} := \bigoplus_{i=1}^{r} \bigoplus_{k=k_0}^{k_1} \mathcal{D}_X[t\partial_t]^r S_k e_i,$$

where $S_k := \partial_t{}^k$ if $k \geq 0$, and $S_k := t^{-k}$ if $k < 0$. Let P be a section of $F_m(\mathcal{D}_{\widetilde{X}})^r$. Then we can write P uniquely in the form

$$P = \sum_{i=1}^{r} \sum_{k=-\infty}^{m} P_{ik}(t\partial_t, x, \partial) S_k e_i \tag{5.1}$$

with $P_{ik} \in \mathcal{D}_X[t\partial_t]$. Then we define

$$\tau(P, k_0) := \sum_{i=1}^{r} \sum_{k=k_0}^{m} P_{ik}(t\partial_t, x, \partial) S_k e_i.$$

Proposition 5.5 *Let* $\mathbf{G}$ *be a finite set of F-involutory generators of* $\mathcal{N}$ *on* X. *Then, for any integers* $k_0 \leq k_1$, *we have an isomorphism*

$$F_{k_1}(\mathcal{M})/F_{k_0-1}(\mathcal{M}) \quad \simeq \quad \widetilde{\mathcal{D}}^{(k_0,k_1)}/\mathcal{G}^{(k_0,k_1)}$$

of left $\mathcal{D}_X[t\partial_t]$*-modules, where* $\mathcal{G}^{(k_0,k_1)}$ *is a left* $\mathcal{D}_X[t\partial_t]$*-module generated by a finite set*

$$\mathbf{G}^{(k_0,k_1)} := \{\tau(S_j P, k_0) \mid P \in \mathbf{G},\ j \in \mathbf{Z},\ k_0 \leq j + \mathrm{ord}_F(P) \leq k_1\}.$$

Let $\chi : \widetilde{\mathcal{D}}^{(k_0+1,k_1+1)} \longrightarrow \widetilde{\mathcal{D}}^{(k_0,k_1)}$ be a left $\mathcal{D}_X[t\partial_t]$-module homomorphism defined by

$$\chi\left(\sum_{i=1}^{r} \sum_{k=k_0}^{k_1} P_{i,k+1}(t\partial_t, x, \partial) S_{k+1} e_i\right) = \sum_{i=1}^{r} \sum_{k=k_0}^{k_1} P_{i,k+1}(t\partial_t - 1, x, \partial) T_k e_i$$

with

$$T_k := \begin{cases} S_k & (k \leq -1) \\ t\partial_t S_k & (k \geq 0). \end{cases}$$

Theorem 5.6 *Under the same assumptions as in Proposition 5.2, we have an isomorphism*

$$\mathcal{H}^{-1}(\mathcal{M}_X^{\bullet}) \ \simeq \ \chi^{-1}(\mathcal{G}^{(k_0,k_1)})/\mathcal{G}^{(k_0+1,k_1+1)}$$

as left $\mathcal{D}_X[t\partial_t]$*-modules. Moreover,* $\chi^{-1}(\mathcal{G}^{(k_0,k_1)})/\mathcal{G}^{(k_0+1,k_1+1)}$ *is a coherent left* $\mathcal{D}_X$*-module.*

A presentation of $\mathcal{H}^{-1}(\mathcal{M}_X^{\bullet})$ as a left coherent $\mathcal{D}_X$-module can be obtained by the following algorithm. Put

$$A^{(k_0,k_1)} := \bigoplus_{i=1}^{r} \bigoplus_{k=k_0}^{k_1} A_n[t\partial_t] S_k e_i.$$

We regard $A^{(k_0,k_1)}$ as a free left $A_n[t\partial_t]$-module of rank $k_1 - k_0 + 1$.

Algorithm 5.7 Input: a finite set $\mathbf{G} \subset (A_{n+1})^r$ of F-involutory generators of $\mathcal{N}$ on X, and integers k_0, k_1 satisfying the assumption of Proposition 5.2.

(1) Let N_1 be the left $A_n[t\partial_t, z]$-submodule of

$$A^{(k_0,k_1)}[z] := \bigoplus_{i=1}^{r} \bigoplus_{k=k_0}^{k_1} A_n[t\partial_t, z] S_k e_i$$

which is generated by

$$\bigcup_{i=1}^{r} \bigcup_{k=k_0}^{k_1} \{(1-z) T_k e_i\} \cup \{zP \mid P \in \mathbf{G}^{(k_0,k_1)}\}$$

with an indeterminate z.

(2) Let $\mathbf{G}_1$ be a Gröbner basis of N_1 with respect to a well-order $\prec_z$ on $L \times \{1, \dots, r\}$ for eliminating z, i.e., satisfying $(\mu, \nu, \alpha, \beta, i) \prec_z (\mu', \nu', \alpha', \beta', j)$ whenever $\mu < \mu'$; here $(\mu, \nu, \alpha, \beta, i) \in L \times \{1, \dots, r\}$ corresponds to the monomial $z^\mu s^\nu x^\alpha \partial^\beta e_i$ with $s = t\partial_t$.

(3) Each element P of $\mathbf{G}_1 \cap A^{(k_0,k_1)}$ can be written uniquely in the form

$$P = \sum_{i=1}^{r} \sum_{k=k_0}^{k_1} Q_{ik}(t\partial_t) T_k e_i$$

with $Q_{ik}(t\partial_t) \in A_n[t\partial_t]$. Then we define $\chi^{-1}(P) \in A^{(k_0+1,k_1+1)}$ by

$$\chi^{-1}(P) := \sum_{i=1}^{r} \sum_{k=k_0}^{k_1} Q_{ik}(t\partial_t + 1) S_{k+1} e_i.$$

Put

$$\mathbf{G}_2 := \{\chi^{-1}(P) \mid P \in \mathbf{G}_1 \cap A^{(k_0,k_1)}\}.$$

Then $\mathbf{G}_2$ generates the left $\mathcal{D}_X[t\partial_t]$-module $\chi^{-1}(\mathcal{G}^{(k_0,k_1)})$.

(4) Suppose $\mathbf{G}_2 = \{P_1, \dots, P_d\}$ and $\mathbf{G}^{(k_0+1,k_1+1)} = \{P_{d+1}, \dots, P_\ell\}$ and put

$$S := \{(Q_1, \dots, Q_\ell) \in A_n[t\partial_t]^\ell \mid \sum_{j=1}^{\ell} Q_j P_j = 0\}.$$

Compute a set of generators $\mathbf{G}_3$ of S by means of a Gröbner basis. Let $\pi_d : A_n[t\partial_t]^\ell \longrightarrow A_n[t\partial_t]^d$ be the projection to the first d components. Then we have an isomorphism

$$\chi^{-1}(\mathcal{G}^{(k_0,k_1)})/\mathcal{G}^{(k_0+1,k_1+1)} \simeq \mathcal{D}_X[t\partial_t]^d / (\mathcal{D}_X[t\partial_t] \otimes_{A_n[t\partial_t]} \pi_d(S))$$

of left $\mathcal{D}_X[t\partial_t]$-modules and $\mathcal{D}_X[t\partial_t] \otimes_{A_n[t\partial_t]} \pi_d(S)$ is generated by $\pi_d(\mathbf{G}_3)$.

(5) Put $L_0 := \mathbf{N}^{1+2n}$ and let $\prec_s$ be a well-order on $L_0 \times \{1, \dots, d\}$ for eliminating s, where $(\mu, \alpha, \beta, i) \in L_0 \times \{1, \dots, d\}$ corresponds to $s^\mu x^\alpha \partial^\beta e'_i$ with $s = t\partial_t$ and $e'_1 = (1, 0, \dots, 0), \dots, e'_d = (0, \dots, 0, 1) \in \mathbf{Z}^d$. Let $\mathbf{G}_4$ be a Gröbner basis of $\pi_d(S)$ with respect to $\prec_s$. At this stage, we have $\mathcal{H}^{-1}(\mathcal{M}_X^\bullet) = 0$ if and only if there exists $P \in \mathbf{G}_4$ whose leading exponent with respect to $\prec_s$ is $(0, i) \in L_0 \times \{1, \dots, d\}$ for each $i = 1, \dots, d$.

(6) For an element P of $(A_n[s])^d$ of the form

$$P = \sum_{i=1}^{d} \sum_{\mu,\alpha,\beta} a_{\mu\alpha\beta i} s^\mu x^\alpha \partial^\beta e'_i,$$

we put

$$\begin{aligned} \deg(P, s) &:= \max\{\mu \in \mathbf{N} \mid a_{\mu\alpha\beta i} \neq 0 \text{ for some } \alpha, \beta \in \mathbf{N}^n,\ i \in \{1, \dots, d\}\}, \\ \operatorname{lcoef}(P, s) &:= \sum_{i=1}^{d} \sum_{\alpha,\beta} a_{m\alpha\beta i} x^\alpha \partial^\beta e'_i \quad \in (A_n)^d \end{aligned}$$

with $m := \deg(P, s)$. Let $\prec'_D$ be a well-order on $\mathbf{N}^{2n} \times \{1, \dots, d\}$ for eliminating ∂, where $(\alpha, \beta, i) \in \mathbf{N}^{2n} \times \{1, \dots, d\}$ corresponds to $x^\alpha \partial^\beta e'_i$. For each $m \in \mathbf{N}$, let $\mathbf{H}_m$ be a Gröbner basis with respect to $\prec'_D$ of the left submodule of $(A_n)^d$ generated by

$$\{\operatorname{lcoef}(P, s) \mid P \in \mathbf{G}_4,\ \deg(P, s) \leq m\}.$$

(7) Put $\mathbf{H}_{m0} := \mathbf{H}_m \cap (K[x])^d$ and

$$U_m := \{p \in X \mid \operatorname{rank}[h(p) \mid h \in \mathbf{H}_{m0}] = d\},$$

where $[h(p) \mid h \in \mathbf{H}_{m0}]$ denotes the matrix consisting of the row vectors $h(p)$ with $h \in \mathbf{H}_{m0}$. Then we have $U_0 \subset U_1 \subset U_2 \subset \ldots$ and $U_m = X$ for some $m \in \mathbf{N}$. On U_m, we have an isomorphism

$$(\mathcal{D}_X[t\partial_t])^d/(\mathcal{D}_X[t\partial_t] \otimes_{A_n[t\partial_t]} \pi_d(S)) \simeq (\mathcal{D}_X[t\partial_t]^{(m)})^d/\mathcal{N}_m$$

of left $\mathcal{D}_X$-modules, where

$$(\mathcal{D}[t\partial_t]^{(m)})^d := \bigoplus_{i=1}^{d}\bigoplus_{\mu=0}^{m-1} \mathcal{D}_X(t\partial_t)^\mu e_i',$$

and $\mathcal{N}_m$ is the left $\mathcal{D}_X$-submodule of $(\mathcal{D}[t\partial_t]^{(m)})^d$ generated by $\{P \in \mathbf{G}_4 \mid \deg(P,s) \leq m-1\}$.

6 Algebraic local cohomology groups

In this section, let X be a Zariski open set of K^n and put $\widetilde{X} := K \times X$. We identify X with the subset $\{0\} \times X$ of K^{n+1} as in the preceding sections. In the sequel we consider a $\mathcal{D}_X$-module $\mathcal{M}$ instead of a $\mathcal{D}_{\widetilde{X}}$-module. Let N be a left A_n-submodule of $(A_n)^r$ and put $M := (A_n)^r/N$ and $\mathcal{M} := \mathcal{D}_X \otimes_{A_n} M$. Then we have $\mathcal{M} = (\mathcal{D}_X)^r/\mathcal{N}$ with $\mathcal{N} := \mathcal{D}_X N$.

Let $f = f(x) \in K[x]$ be a non-constant polynomial and put $Y := \{x \in X \mid f(x) = 0\}$. Then the algebraic local cohomology group $\mathcal{H}^j_{[Y]}(\mathcal{M})$ has a structure of left $\mathcal{D}_X$-module and vanishes for $j \neq 0, 1$ ([8]). Our purpose is to give an algorithm of computing $\mathcal{H}^j_{[Y]}(\mathcal{M})$ as a left $\mathcal{D}_X$-module. In general, for an $\mathcal{O}_X$-module $\mathcal{F}$, put

$$\Gamma_{[Y]}(\mathcal{F}) := \{u \in \mathcal{F} \mid f^k u = 0 \quad \text{for some } k \in \mathbf{N}\}.$$

Then $\mathcal{H}^j_{[Y]}(\mathcal{F})$ is defined as the j-th derived functor of $\Gamma_{[Y]}$.

Put $Z := \{(t,x) \in K \times X \mid t - f(x) = 0\}$. Let $\mathcal{J}_Z$ be a left ideal of $\mathcal{D}_{\widetilde{X}}$ generated by $t - f(x)$, $\partial_1 + (\partial f/\partial x_1)\partial_t$, $\ldots$, $\partial_n + (\partial f/\partial x_n)\partial_t$, and put $\mathcal{B}_{[Z]} := \mathcal{D}_{\widetilde{X}}/\mathcal{J}_Z$. We denote by $\delta(t-f)$ the residue class of $1 \in \mathcal{D}_{\widetilde{X}}$ in $\mathcal{B}_{[Z]}$.

Put $\mathcal{L} := \mathcal{O}_X[f^{-1}, s]f^s$, where f^s is regarded as a free generator. Then $\mathcal{L}$ has a natural structure of left $\mathcal{D}_X[s]$-module. As was observed by Malgrange [14], $\mathcal{L}$ has a structure of left $\mathcal{D}_{\widetilde{X}}$-module so that

$$t(g(s)f^s) = g(s+1)f^{s+1}, \quad \partial_t(g(s)f^s) = -sg(s-1)f^{s-1} \tag{6.1}$$

for $g(s) \in \mathcal{O}_X[f^{-1}, s]$. This implies that there exists an injective homomorphism $\iota : \mathcal{B}_{[Z]}|_X \longrightarrow \mathcal{L}$ of left $\mathcal{D}_{\widetilde{X}}$-modules such that $\iota(\delta(t-f)) = f^s$ ([14]).

Lemma 6.1 *We have an isomorphism $(\mathcal{B}_{[Z]})^{\bullet}_X \simeq \mathbf{R}\Gamma_{[Y]}(\mathcal{O}_X)[1]$ in the derived category of left $\mathcal{D}_X$-modules, where $\mathbf{R}\Gamma_{[Y]}$ denotes the right derived functor of $\Gamma_{[Y]}$, and $[1]$ the translation functor ([6]).*

Now let $\pi : \widetilde{X} \longrightarrow X$ be the projection. Then the tensor product $\mathcal{B}_{[Z]} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}$ has a structure of sheaves of left $\mathcal{D}_{\widetilde{X}}$-modules. Let π_1 and π_2 be the projections of $\widetilde{X} \times X$ to $\widetilde{X}$ and to X respectively defined by $\pi_1(t,x,y) = (t,x)$ and $\pi_2(t,x,y) = y$ for $t \in K$ and $x, y \in X$. Put

$$\Delta := \{(t,x,y) \in \widetilde{X} \times X \mid x = y\}$$

and

$$\mathcal{D}_{\Delta \to \widetilde{X}\times X} := \mathcal{D}_{\widetilde{X}\times X}/((x_1 - y_1)\mathcal{D}_{\widetilde{X}\times X} + \cdots + (x_n - y_n)\mathcal{D}_{\widetilde{X}\times X}).$$

Lemma 6.2 *Let $\mathcal{F}$ be a left $\mathcal{D}_{\widetilde{X}}$-module. Then we have*

$$\mathcal{F} \overset{\mathbf{L}}{\otimes}_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M} \simeq \mathcal{D}_{\Delta \to \widetilde{X}\times X} \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_{\widetilde{X}\times X}} (\mathcal{F}\hat{\otimes}\mathcal{M})$$

with

$$\mathcal{F}\hat{\otimes}\mathcal{M} := \mathcal{D}_{\widetilde{X}\times X} \otimes_{\pi_1^{-1}\mathcal{D}_{\widetilde{X}} \otimes \pi_2^{-1}\mathcal{D}_X} (\pi_1{}^{-1}\mathcal{F} \otimes_K \pi_2{}^{-1}\mathcal{M}).$$

Lemma 6.3 *The i-th torsion group $\mathcal{T}or_i^{\pi^{-1}\mathcal{O}_X}(\mathcal{B}_{[Z]}, \pi^{-1}\mathcal{M})$ vanishes for $i \neq 0$.*

Theorem 6.4 *We have isomorphisms*

$$\mathcal{H}^j((\mathcal{B}_{[Z]} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M})^\bullet_X) \simeq \mathcal{H}^{j+1}_{[Y]}(\mathcal{M})$$

of left $\mathcal{D}_X$-modules for $j = -1, 0$.

In what follows, we shall denote $\mathcal{F} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}$ by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}$ for a $\mathcal{D}_{\widetilde{X}}$-module $\mathcal{F}$. In view of Theorems 5.6, 5.8, 5.10 and 6.3, we obtain an algorithm for computing the algebraic local cohomology groups $\mathcal{H}^j_{[Y]}(\mathcal{M})$ for $j = 0, 1$ if there is an algorithm for computing $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ as a left $\mathcal{D}_{\widetilde{X}}$-module. In fact, this tensor product can be computed as follows:

Lemma 6.5 *Let $\mathcal{J}_Z$ be as above. Then we have an isomorphism $\mathcal{B}_{[Z]}\hat{\otimes}\mathcal{M} \simeq (\mathcal{D}_{\widetilde{X}\times X})^r/\mathcal{N}_Z$ with $\mathcal{N}_Z := \mathcal{J}_Z\hat{\otimes}(\mathcal{D}_X)^r + \mathcal{D}_{\widetilde{X}}\hat{\otimes}\mathcal{N}$.*

For $i = 1, \ldots, n$, put

$$\Delta_i := \{(t,x,y) \in \widetilde{X} \times X \mid x_j = y_j \text{ for } j = 1, \ldots, i\}.$$

Then we have

$$\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M} \simeq (\ldots((\mathcal{B}_{[Z]}\hat{\otimes}\mathcal{M})_{\Delta_1})_{\Delta_2} \cdots)_{\Delta_n}$$

by virtue of Lemma 6.2. Since Δ_i is non-characteristic for $\mathcal{B}_{[Z]}\hat{\otimes}\mathcal{M}$ in view of the proof of Lemma 6.3, we can compute $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ by applying Theorem 5.7 repeatedly with $k_0 = k_1 = 0$.

Lemma 6.6 *If $\mathcal{M}$ is holonomic, then $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ is specializable along X.*

Thus we have obtained an algorithm for computing $\mathcal{H}^j_{[Y]}(\mathcal{M})$ $(j = 0, 1)$ by applying Theorem 5.7 and Algorithm 5.10 to $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ under the condition that $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ is specializable along X. In particular, we have proved the following statement effectively:

Corollary 6.7 *If $\mathcal{B}_{[Z]}\otimes_{\mathcal{O}_X}\mathcal{M}$ is specializable along X, then $\mathcal{H}^j_{[Z]}(\mathcal{M})$ $(j=0,1)$ are coherent left $\mathcal{D}_X$-modules.*

Let us describe $\mathcal{H}^1_{[Y]}(\mathcal{M})$ more concretely. First note that $\mathcal{H}^1_{[Y]}(\mathcal{M}) \simeq \mathcal{M}[f^{-1}]/\mathcal{M}$ with $\mathcal{M}[f^{-1}] := \mathcal{O}_X[f^{-1}]\otimes_{\mathcal{O}_X}\mathcal{M}$. By applying Theorem 5.7 to $\mathcal{B}_{[Z]}\otimes_{\mathcal{O}_X}\mathcal{M}$, we know that $\mathcal{M}[f^{-1}]/\mathcal{M}$ is generated by the modulo classes $v_{ik} := [f^{-k}\otimes u_i]$ in $(\mathcal{O}[f^{-1}]\otimes_{\mathcal{O}_X}\mathcal{M})/\mathcal{M}$ with $k_0 \le k \le k_1$ and $1 \le i \le r$, and the relations among the generators $k!v_{ik}$ are given by $\mathcal{N}_X$ of Theorem 5.7. Actually, v_{ik_1} with $1 \le i \le r$ generate $\mathcal{M}[f^{-1}]/\mathcal{M}$ and the relations among these generators can be obtained by eliminating v_{ik} with $k < k_1$.

Our next aim is to give an algorithm of computing the b-function for a polynomial f and a section u of $\mathcal{M}$. Put $\mathcal{M}[s] := K[s]\otimes_K \mathcal{M}$. Then we have

$$\mathcal{L}\otimes_{\mathcal{O}_X[s]}\mathcal{M}[s] = \mathcal{L}\otimes_{\mathcal{O}_X[s]}(\mathcal{O}_X[s]\otimes_{\mathcal{O}_X}\mathcal{M}) = \mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{M}.$$

Note that an arbitrary element of $\mathcal{L}\otimes_{\mathcal{O}_X[s]}\mathcal{M}[s]$ can be expressed in the form $f^{s-m}\otimes u$ with some $m\in\mathbf{N}$ and $u\in\mathcal{M}[s]$.

Lemma 6.8 *Let u be a section of $\mathcal{M}[s]$ and let m be a nonnegative integer. Then we have $f^{s-m}\otimes u = 0$ in $\mathcal{L}\otimes_{\mathcal{O}_X[s]}\mathcal{M}[s]$ if and only if $f^k u = 0$ holds in $\mathcal{M}[s]$ with some $k\in\mathbf{N}$.*

Let u be a section of $\mathcal{M}$ and P a section of $\mathcal{D}_X[s]$. Then the identity $P(f^s u) = 0$ means by definition that there exists $m\in\mathbf{N}$ so that $Q := f^{m-s}Pf^s$ is contained in $\mathcal{D}_X[s]$ and that $Qu = 0$ holds in $\mathcal{M}[s]$ (cf. [8]).

Lemma 6.9 *For $u\in\mathcal{M}$ and $P\in\mathcal{D}_X[s]$, we have $P(f^s u) = 0$ if and only if $P(f^s\otimes u) = 0$ in $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{M}$.*

Lemma 6.10 *$\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$ if and only if $f : \mathcal{B}_{[Z]}\otimes_{\mathcal{O}_X}\mathcal{M} \longrightarrow \mathcal{B}_{[Z]}\otimes_{\mathcal{O}_X}\mathcal{M}$ is injective.*

Lemma 6.11 *Let p be a point of Y. Then any germ v of $\mathcal{B}_{[Z]}\otimes_{\mathcal{O}_X}\mathcal{M}$ at p is uniquely written in the form*

$$v = \sum_{i=0}^{k} {\partial_t}^i\delta(t-f)\otimes u_i \tag{6.2}$$

with $u_i\in\mathcal{M}_p$ and $k\in\mathbf{N}$.

Proposition 6.12 *The homomorphism*

$$\iota\otimes 1 \; : \; \mathcal{B}_{[Z]}\otimes_{\mathcal{O}_X}\mathcal{M} \longrightarrow \mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{M}$$

is injective if and only if $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$.

Theorem 6.13 *Assume $r = 1$ and let $u\in\mathcal{M}$ be the residue class of $1\in\mathcal{D}_X$. Let $b_X(s)$ be the b-function of $\mathcal{B}_{[Z]}\otimes_{\mathcal{O}_X}\mathcal{M}$ along X with respect to the filtration $\{F_k(\mathcal{D}_{\tilde{X}})(\delta(t-f)\otimes u)\}_{k\in\mathbf{Z}}$ and let $b(s)$ be the b-function for f and u defined by (1.1), both at a point p of Y. Then we have the following:*

(1) $b(s)$ divides $b_X(-s-1)$;

(2) if $\mathcal{H}^0_{[Y]}(\mathcal{M})_p = 0$, then we have $b(s) = \pm b_X(-s-1)$;

(3) A nonzero b-function $b(s)$ for f and u exists at $p \in X$ if and only if $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ is specializable along X at p.

Thus we have obtained an algorithm for computing the b-function for f and $u \in \mathcal{M}$ under the assumption $\mathcal{H}^0_{[Y]}(\mathcal{D}_X u) = 0$, which can be determined by Algorithm 5.10. Note that we do not need this assumption for deciding whether a nonzero b-function exists. This generalizes an algorithm of computing the Bernstein-Sato polynomial given in [19].

Example 6.14 Put $\mathcal{M} := \mathcal{H}^1_{[Y]}(\mathcal{O}_X)$ and u be the residue class of f^{-1} in $\mathcal{M} = \mathcal{O}_X[f^{-1}]/\mathcal{O}_X$. Let p be a point of Y. Then the b-function for f and u at p is 1 since $fu = 0$ in $\mathcal{M}$. On the other hand, the b-function of $\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M}$ along X at p is $b_X(s) = s+1$. In fact, since $t(\delta(t-f) \otimes u) = \delta(t-f) \otimes (fu) = 0$, we know that $b_X(s)$ divides $s+1$. If $b_X(s) = 1$, then we should have

$$\mathcal{M} = \mathcal{H}^0_{[Y]}(\mathcal{M}) \simeq \mathcal{H}^{-1}((\mathcal{B}_{[Z]} \otimes_{\mathcal{O}_X} \mathcal{M})^\bullet_X) = 0$$

by virtue of Proposition 5.2 and Theorem 6.4, which is a contradiction.

It is also possible (in generic cases) to compute $\mathcal{H}^j_{[Y]}(\mathcal{M})$ for algebraic set Y of codimension greater than one. For example, let $f_1(x), f_2(x)$ be two polynomials and put

$$\begin{aligned} Y_i &:= \{x \in X \mid f_i(x) = 0\} \quad (i = 1, 2), \\ Y &:= Y_1 \cap Y_2. \end{aligned}$$

Assume that $\mathcal{H}^j_{[Y_1]}(\mathcal{M}) = 0$ for $j \neq j_0$. Then we can compute

$$\mathcal{H}^j_{[Y]}(\mathcal{M}) = \mathcal{H}^{j-j_0}_{[Y_2]}(\mathcal{H}^{j_0}_{[Y_1]}(\mathcal{M}))$$

explicitly by applying the above method first to f_1 and $\mathcal{M}$, then to f_2 and $\mathcal{H}^{j_0}_{[Y_1]}(\mathcal{M})$.

Example 6.15 Put $X = K^3$, $f_1 := x^2 - y^3$, $f_2 := y^2 - z^3$, and consider the space curve $Y := \{(x,y,z) \in X \mid f_1(x,y,z) = f_2(x,y,z) = 0\}$. Then we have $\mathcal{H}^j_{[Y]}(\mathcal{O}_X) = 0$ for $j \neq 2$ and

$$\mathcal{H}^2_{[Y]}(\mathcal{O}_X) \simeq \mathcal{D}_X/\mathcal{I},$$

where $\mathcal{I}$ is the left ideal of $\mathcal{D}_X$ generated by f_1, f_2 and

$$9x\partial_x + 6y\partial_y + 4z\partial_z + 30, \qquad 9y^2z^2\partial_x + 6xz^2\partial_y + 4xy\partial_z.$$

Let u_j be the residue class of $f_j{}^{-1}$ in $\mathcal{H}^1_{[Y_j]}(\mathcal{O}_X) = \mathcal{O}_X[f_j{}^{-1}]/\mathcal{O}_X$ with $Y_j := \{(x,y,z) \mid f_j(x,y,z) = 0\}$. Then the b-function for f_2 and u_1 is

$$(s+1)\left(s+\frac{1}{12}\right)\left(s+\frac{5}{12}\right)\left(s+\frac{7}{12}\right)\left(s+\frac{5}{6}\right)\left(s+\frac{11}{12}\right)\left(s+\frac{7}{6}\right)$$

at $(0,0,0)$, and $s+1$ on $Y \setminus \{(0,0,0)\}$. The b-function for f_1 and u_2 is

$$(s+1)\left(s+\frac{7}{18}\right)\left(s+\frac{11}{18}\right)\left(s+\frac{13}{18}\right)\left(s+\frac{5}{6}\right)\left(s+\frac{17}{18}\right)\left(s+\frac{19}{18}\right)\left(s+\frac{7}{6}\right)\left(s+\frac{23}{18}\right)$$

at $(0,0,0)$, and $s+1$ on $Y \setminus \{(0,0,0)\}$.

7 Localization of a D-module

We retain the notation of the preceding section. Our primary goal in this section is to obtain an algorithm for computing the localization $\mathcal{M}[f^{-1}] := \mathcal{O}_X[f^{-1}]\otimes_{\mathcal{O}_X}\mathcal{M}$ as a left $\mathcal{D}_X$-module under the assumption $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$. For this purpose, we shall first compute

$$\mathcal{P} := \mathcal{D}_X[s](f^s \otimes u_1) + \ldots + \mathcal{D}_X[s](f^s \otimes u_r),$$

which is a left $\mathcal{D}_X[s]$-submodule of $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{M}$, and then specialize the parameter s.

Proposition 7.1 *Assume $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$. Then there is an algorithm to compute a set of generators on X of the left $\mathcal{D}_X[s]$-module*

$$\mathcal{Q} := \{(Q_1, \ldots, Q_r) \in (\mathcal{D}_X[s])^r \mid \sum_{i=1}^{r} Q_i(s)(f^s \otimes u_i) = 0\}.$$

Now let us fix an arbitrary element s_0 of K and consider the specialization $s = s_0$ of the parameter s. Put $\mathcal{L}(s_0) := \mathcal{O}_X[f^{-1}]f^{s_0}$, where f^{s_0} is regarded as a free generator. Let $\rho : \mathcal{L} \longrightarrow \mathcal{L}(s_0)$ be the surjective homomorphism of left $\mathcal{D}_X$-modules defined by $\rho(g(s,x)f^{s-m}) = g(s_0,x)f^{s_0-m}$ for $g(s,x) \in \mathcal{O}_X[s, f^{-1}]$ and $m \in \mathbf{N}$. Then it is easy to see that ρ induces an isomorphism $\mathcal{L}(s_0) \simeq \mathcal{L}/(s - s_0)\mathcal{L}$ as left $\mathcal{D}_X$-modules.

Since the proof of Lemma 6.8 is also valid with s specialized to an element of K, we get the following:

Lemma 7.2 *Let u be a section of $\mathcal{M}$ and let m be a nonnegative integer. Fix $s_0 \in K$. Then we have $f^{s_0-m} \otimes u = 0$ in $\mathcal{L}(s_0) \otimes_{\mathcal{O}_X} \mathcal{M}$ if and only if $f^k u = 0$ holds in $\mathcal{M}$ with some $k \in \mathbf{N}$.*

Consider the homomorphism

$$\rho \otimes 1 \; : \; \mathcal{L} \otimes_{\mathcal{O}_X[s]} \mathcal{M}[s] = \mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{M} \longrightarrow \mathcal{L}(s_0)\otimes_{\mathcal{O}_X}\mathcal{M}$$

and put $\mathcal{P}(s_0) := (\rho \otimes 1)(\mathcal{P})$. Our aim is to obtain an algorithm of computing $\mathcal{P}(s_0)$. Since $(s - s_0)\mathcal{P}$ is contained in the kernel of $\rho \otimes 1$, there exists a surjective homomorphism $\mathcal{P}/(s - s_0)\mathcal{P} \longrightarrow \mathcal{P}(s_0)$ induced by $\rho \otimes 1$. A sufficient condition for this homomorphism to be an isomorphism is given as follows (cf. Proposition 6.2 of [7] for the case $\mathcal{M} = \mathcal{O}_X$).

Proposition 7.3 *Assume that the b-function $b_i(s,p)$ for f and u_i at $p \in X$ exists for $i = 1, \ldots, r$. Assume, moreover, that $b_i(s_0 - \nu) \neq 0$ for any $i = 1, \ldots, r$, $\nu = 1, 2, 3, \ldots$, and $p \in Y$. Then the homomorphism $\mathcal{P}/(s-s_0)\mathcal{P} \longrightarrow \mathcal{P}(s_0)$ is a left $\mathcal{D}_X$-module isomorphism. In particular, we have an isomorphism $\mathcal{P}(s_0) \simeq (\mathcal{D}_X)^r/\mathcal{Q}(s_0)$ with $\mathcal{Q}(s_0) := \{Q(s_0) \mid Q(s) \in \mathcal{Q}\}$.*

Thus we have obtained an algorithm for computing $\mathcal{P}(s_0)$ under the conditions of the above proposition. Note that it amounts to computing $\mathcal{L}(s_0)\otimes_{\mathcal{O}_X}\mathcal{M}$ as follows.

Proposition 7.4 *Under the same assumptions as in the preceding proposition, we have $\mathcal{P}(s_0) = \mathcal{L}(s_0)\otimes_{\mathcal{O}_X}\mathcal{M}$.*

Proposition 7.5 *Assume that $\mathcal{B}_{[Z]}\otimes_{\mathcal{O}_X}\mathcal{M}$ is specializable along X. Then there exists a positive integer k_0 so that $\mathcal{M}[f^{-1}]$ is isomorphic to $(\mathcal{D}_X)^r/\mathcal{Q}(-k)$ as left $\mathcal{D}_X$-module for any integer $k \geq k_0$.*

Thus under the condition that $\mathcal{B}_{[Z]}\otimes\mathcal{M}$ is specializable along X and that $\mathcal{H}^0_{[Y]}(\mathcal{M}) = 0$, we have obtained an algorithm of computing $\mathcal{M}[f^{-1}]$ combining Propositions 7.1 and 7.5. More concretely, we have

$$\mathcal{M}[f^{-1}] = \sum_{i=1}^{r} \mathcal{D}_X(f^{-k_0}\otimes u_i),$$

and our algorithm computes a finite subset of $(A_n)^r$ which generates the left $\mathcal{D}_X$-module

$$\mathcal{Q}(-k_0) = \{P \in \mathcal{D}_X \mid \sum_{i=1}^{r} P_i(f^{-k_0}\otimes u_i) = 0\}$$

on X. In particular, by applying the above argument to $\mathcal{M} := \mathcal{D}_X g^{s_2}$ with another polynomial $g \in K[s]$ and a constant $s_2 \in K$, we obtain an algorithm for computing $\mathcal{D}_X(f^{s_1}f^{s_2})$ for generic $s_1, s_2 \in K$ as follows: First, we can compute $\mathcal{D}_X g^{s_2}$ if the Bernstein-Sato polynomial $b_g(s)$ of g satisfies $b_g(s_2-\nu)\neq 0$ for $\nu = 1,2,3,\ldots$ (cf. [21]). Then we have

$$(\mathcal{D}_X f^{s_1})\otimes_{\mathcal{O}_X}(\mathcal{D}_X g^{s_2}) \;\simeq\; \mathcal{D}_X(f^{s_1}g^{s_2})$$

by virtue of Lemma 7.2, where $\mathcal{D}_X(f^{s_1}g^{s_2})$ is the left $\mathcal{D}_X$-submodule of $\mathcal{O}_X[f^{-1},g^{-1}]f^{s_1}g^{s_2}$ generated by $f^{s_1}g^{s_2}$. Thus by applying the arguments in this section, we can compute $\mathcal{D}_X(f^{s_1}g^{s_2})$ if, in addition to the above condition, the b-function $b_{12}(s)$ for f and g^{s_2} satisfies $b_{12}(s_0-\nu)\neq 0$ for $\nu = 1,2,3,\ldots$. Note that we always have $\mathcal{H}^0_{[Y]}(\mathcal{D}_X g^{s_2}) = 0$.

Hence by choosing positive integers k_1, k_2 so that $s_1 = -k_1$ and $s_2 = -k_2$ satisfy the above conditions, we get an algorithm to compute the localization $\mathcal{O}_X[f^{-1},g^{-1}] = \mathcal{O}_X[f^{-k_1},g^{-k_2}]$ as $\mathcal{D}_X$-module.

If we regard s_1, s_2 as inderterminates not as constants, then it is also interesting to consider the left $\mathcal{D}_X[s_1,s_2]$-module $\mathcal{D}_X[s_1,s_2]f^{s_1}g^{s_2}$. An algorithm for computing this module can be obtained by generalizing a method used in [21], or also by modifying the arguments in this section so as to be adapted to the case where $\mathcal{M}$ is a $\mathcal{D}_X[s_2]$-module. We shall discuss this problem elsewhere.

Example 7.6 Put $X = K^3 \ni (x,y,z)$ and write $\partial_x := \partial/\partial x, \partial_y := \partial/\partial y, \partial_z := \partial/\partial z$. Put $f_1 := x^2 - y^3$ and $f_2 := y^2 - z^3$. Let $s_1, s_2 \in K$ be constants. The Bernstein-Sato polynomial of f_2 at the singular point $(0,0,0)$ is $b_2(s) = (s+1)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)$. We have $\mathcal{D}_X f^{s_2} = \mathcal{D}_X/\mathcal{I}$ with the left ideal of $\mathcal{D}_X$ generated by

$$\partial_x, \quad 3y\partial_y + 2z\partial_z - 6s_2, \quad 3z^2\partial_y + 2y\partial_z, \quad (y^2 - z^3)\partial_z + 3z^2 s_2$$

if $b_2(s_2-\nu)\neq 0$ for any $\nu = 1,2,3,\ldots$. Then the b-function for f_1 and $f_2^{s_2}$ is

$$\begin{aligned} b_{12}(s) \;=\;& (s+1)\left(s+\tfrac{5}{6}\right)\left(s+\tfrac{7}{6}\right)\left(s+\tfrac{2}{3}s_2+\tfrac{19}{18}\right)\left(s+\tfrac{2}{3}s_2+\tfrac{23}{18}\right)\\ & \left(s+\tfrac{2}{3}s_2+\tfrac{25}{18}\right)\left(s+\tfrac{2}{3}s_2+\tfrac{29}{18}\right)\left(s+\tfrac{2}{3}s_2+\tfrac{31}{18}\right)\left(s+\tfrac{2}{3}s_2+\tfrac{35}{18}\right)\end{aligned}$$

at $(0,0,0)$; while at the other points we have

$$b_{12}(s)=\begin{cases}(s+1)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right) & \text{on } \{(0,0,z)\mid z\neq 0\},\\ s+1 & \text{on } \{(x,y,z)\mid x^2-y^3=0,\ yz\neq 0\},\\ 1 & \text{on } \{(x,y,z)\mid x^2-y^3\neq 0\}.\end{cases}$$

If s_1 satisfies $b_{12}(s_1-\nu)\neq 0$ for any $\nu=1,2,3,\ldots$ in addition to the above condition on s_2.. Under the same assumptions, we have $\mathcal{D}_X(f_1^{s_1}f_2^{s_2})=\mathcal{D}_X/\mathcal{I}(s_1,s_2)$ with the left ideal $\mathcal{I}(s_1,s_2)$ of $\mathcal{D}_X$ generated by

$$\begin{cases}9x\partial_x+6y\partial_y+4z\partial_z-6(3s_1+2s_2),\\ (y^2-z^3)\partial_z+3z^2s_2,\\ (x^2-y^3)\partial_x-2s_1x,\\ 9y^2z^2\partial_x+6xz^2\partial_y+4xy\partial_z,\\ 3y(x^2-y^3)\partial_y+2z(x^2-y^3)\partial_z+3(-2s_2x^2+(3s_1+2s_2)y^3),\\ 3z^2(x^2-y^3)\partial_y+2y(x^2-y^3)\partial_z+9s_1y^2z^2.\end{cases}$$

In particular the above assumptions are satisfied for $s_1=s_2=-1$. Hence we have $\mathcal{O}_X[f_1^{-1},f_2^{-1}]\simeq\mathcal{D}_X/\mathcal{I}(-1,-1)$. By regarding s_1,s_2 as indeterminates not as constants, we have also $\mathcal{D}_X[s_1,s_2](f_1^{s_1}f_2^{s_2})=\mathcal{D}_X[s_1,s_2]/\mathcal{I}(s_1,s_2)$. Then we can verify by elimination that the ideal $(\mathcal{I}(s_1,s_2)+\mathcal{D}_X[s_1,s_2]f_1f_2)_0\cap K[s_1,s_2]$ of $K[s_1,s_2]$ is generated by a single element

$$\begin{aligned}b(s_1,s_2)\ :=\ &(s_1+1)(6s_1+5)(6s_1+7)(s_2+1)(6s_2+5)(6s_2+7)(\ell+19)(\ell+23)\\ &(\ell+25)(\ell+29)(\ell+31)(\ell+35)(\ell+37)(\ell+41)(\ell+43)(\ell+47)\end{aligned}$$

with $\ell:=18s_1+12s_2$. This means that $b(s_1,s_2)$ is a minimum polynomial that satisfies a functional equation of the form $P(f_1^{s_1+1}f_2^{s_2+1})=b(s_1,s_2)f_1^{s_1}f_2^{s_2}$ with some germ P of $\mathcal{D}_X[s_1,s_2]$ at 0 (cf. [22], [15]).

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